

Complex linear spaces and normed linear spaces.

Theorem: - Let N be a normed linear space and $x_0 \neq 0$ non-zero vector in N , there exists a functional F in N^* such that

$$F(x_0) = \|x_0\| \text{ and } \|F\| = 1.$$

In particular, if $x \neq y$ ($x, y \in N$), then there exists an $f \in N^*$ such that $f(x) \neq f(y)$.

Proof: - Let $M = \{\alpha x_0\}$ be the linear subspace of N spanned by x_0 . Define f_0 on M by $f_0(\alpha x_0) = \alpha \|x_0\|$. We show that f_0 is a functional on M such that $\|f_0\| = 1$. f_0 is linear.

Let $y_1, y_2 \in M$ so that $y_1 = \alpha x_0, y_2 = \beta x_0$ for some scalars α and β . If γ, δ are any scalars, then

$$\begin{aligned} f_0(\gamma y_1 + \delta y_2) &= f_0(\gamma \alpha x_0 + \delta \beta x_0) = f_0((\gamma \alpha + \delta \beta) x_0) \\ &= (\gamma \alpha + \delta \beta) \|x_0\| \text{ by def. of } f_0 = \gamma \alpha \|x_0\| + \delta \beta \|x_0\| \\ &= \gamma f_0(\alpha x_0) + \delta f_0(\beta x_0) = \gamma f_0(y_1) + \delta f_0(y_2). \end{aligned}$$

f_0 is bounded.

$$\begin{aligned} \text{Let } y = \alpha x_0 \in M \text{ so that } \|y\| = \|\alpha x_0\| = |\alpha| \|x_0\|, \text{ now } |f_0(y)| &= |f_0(\alpha x_0)| = |\alpha| \|x_0\| \\ &= \|y\| < \|y\|. \end{aligned}$$

Hence f_0 is bounded. It follows that f_0 is a functional on M .

$$\begin{aligned} \text{Further, } \|f_0\| &= \sup \{ |f_0(y)| : y \in M, \|y\| = 1 \} \\ &= \sup \{ \|\alpha x_0\| : \alpha x_0 \in M, \|\alpha x_0\| = 1 \} = 1. \end{aligned}$$

Also $f_0(x_0) = \|x_0\|$ by definition of f_0 (choose $\alpha = 1$)

Hence by Hahn-Banach theorem f_0 can be extended to a norm preserving functional $F \in N^*$ so that

$$F(x_0) = f_0(x_0) = \|x_0\| \text{ and } \|F\| = \|f_0\| = 1.$$

In the particular case, since $x \neq y$, $x - y \neq 0$ and so we know that there exists an $f \in N^*$ such that

$$f(x - y) = \|x - y\| \neq 0 \Rightarrow f(x) - f(y) \neq 0 \Rightarrow f(x) \neq f(y)$$

This shows that N^* separates vectors in N .

Theorem: Let M be a closed linear subspace of a normed linear space N and x_0 a vector not in M . Then there exists a functional F in N^* such that $F(M) = \{0\}$ and $F(x_0) \neq 0$.

Proof: Consider the natural map

$$\phi: N \rightarrow N/M : \phi(x) = x+M$$

~~As shown~~ ϕ is a continuous linear transformation and if $m \in M$, then $\phi(m) = m+M = 0$

$$\phi(M) = \{0\} \quad \text{--- (1)}$$

Also since $x_0 \notin M$, we have

$$\phi(x_0) = x_0+M \neq 0 \quad (\text{i.e. } \neq \text{zero vector } M \text{ of } N/M)$$

Hence by ~~the~~ ^{have} there exists a functional $f \in (N/M)^*$

$$\text{such that } f(x_0+M) = \|x_0+M\| \neq 0 \quad \text{--- (2)}$$

[$\because x_0+M \neq \text{zero vector}$]

We now define F by $F(x) = f(\phi(x))$. Then F is a linear functional on N with the desired properties as shown below.

F is linear.

$$F(\alpha x + \beta y) = f(\phi(\alpha x + \beta y)) = f((\alpha x + \beta y) + M)$$

$$= f(\alpha(x+M) + \beta(y+M))$$

$$= \alpha f(x+M) + \beta f(y+M) \quad [\because f \text{ is linear on } N/M]$$

$$= \alpha f(\phi(x)) + \beta f(\phi(y)) = \alpha F(x) + \beta F(y).$$

F is bounded.

$$\|F(x)\| = |f(\phi(x))| \leq \|f\| \|\phi(x)\|$$

$$\leq \|f\| \|\phi\| \|x\| \leq \|f\| \|x\|$$

$$(\because \|\phi\| \leq 1)$$

Since f is bounded (being a functional), it follows from the above inequality that F is bounded. Thus F is a functional on N i.e. $F \in N^*$. Further if $m \in M$, then $F(m) = f(\phi(m)) = f(0) = 0$ [$\because \phi(m) = 0$ by (1)]

And $F(x_0) = f(\phi(x_0)) = f(x_0+M) \neq 0$ by (2).

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